ON THE TENSORIAL PROPERTIES OF THE GENERALIZED JACOBI EQUATION

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Abstract. The generalized Jacobi equation is a differential equation in local coordinates that describes the behavior of infinitesimally close geodesics with an arbitrary relative velocity. In this note we study some transformation properties for solutions to this equation. We prove two results. First, under any affine coordinate changes we show that the tensor transformation rule maps solutions to solutions. As a consequence, the generalized Jacobi equation is a tensor equation when restricted to suitable Fermi coordinate systems along a geodesic. Second, in dimensions $n \geq 3$, we explicitly show that the transformation rule does not in general preserve solutions to the generalized Jacobi equation.

1. Introduction

Suppose M is a manifold with an affine and torsion free connection ∇. In this setting, the *Jacobi equation* is a fundamental equation that describes the qualitative behavior of infinitesimally close geodesics on M. See for example, [LC27] and [Hic65]. One way to derive the Jacobi equation is to consider the deviation $\xi^\mu(s) = x^\mu(s) - X^\mu(s)$ between two neighboring geodesic X and x. Then the Jacobi equation follows by subtracting the geodesic equations for X and x and assuming that $\xi^\mu(s)$ and $\dot{\xi}^\mu(s) = \frac{d\xi^\mu}{ds}(s)$ are both infinitesimal quantities. This is a standard argument. See for example [Per08] and references therein. The *generalized Jacobi equation* is derived in the same way, but under the weaker hypothesis that only $\xi^\mu(s)$ is an infinitesimal quantity. That is, the geodesics are assumed to be infinitesimally close, but their relative velocity $\dot{\xi}^\mu$ does not need to be small. Under this weaker hypothesis, equations for the displacement ξ^μ was derived first in the Lorentzian case by Hodgkinson [Hod72] and independently by Ciufolini [Ciu86]. This generalization of the Jacobi equation have been investigated by several authors, specially with applications on astrophysics and cosmology. See for instance [Mas75, CM02, Per08].

We will work in the setting of an affine and torsion-free connection. In this setting, the generalized Jacobi equation was derived by Perlick [Per08]. In more detail, a collection of n functions $\xi^\mu \colon I \to \mathbb{R}$ along a geodesic $X \colon I \to M$ is a solution to the *generalized Jacobi field* in coordinates $\{x^\mu\}_{\mu=0}^{n-1}$ if

$$\ddot{\xi}^\mu + \Gamma^\mu_{\nu\rho}\left(2\dot{\xi}^\rho \dot{X}^\nu + \dot{\xi}^\rho\dot{\xi}^\nu\right) + \frac{\partial \Gamma^\mu_{\rho\nu}}{\partial x^\tau}\xi^\tau\left(\dot{X}^\rho + \dot{\xi}^\rho\right)\left(\dot{X}^\nu + \dot{\xi}^\nu\right) = 0, \tag{1}$$

where dots indicate ordinary derivatives with respect to the parameter s of the central geodesic $X^\mu(s)$ and $\Gamma^\mu_{\nu\rho}$ are the connection coefficients of ∇.

Let us first observe that in contrast to the usual Jacobi equation, the generalized Jacobi equation is a non-linear equation in the unknown functions ξ^μ. This makes the analysis of the solution space more difficult. For example, the solution space need not be a vector space and in general, there are no (known) results that associate a solution ξ^μ to a geodesic variation as for the Jacobi equation.

The generalized Jacobi equation (1) is an equation in local coordinates. If ∇ is *flat*, there exits coordinates where $\Gamma^\mu_{\nu\rho} = 0$ [Shi07, Proposition 1.1]. Then the generalized Jacobi equation and the usual Jacobi equation both simplify into $\ddot{\xi}^\mu = 0$. More generally, Perlick [Per08] has proven that if X is a lightlike geodesic in a special class of Lorentz metrics, or *planewave metrics*, then there are coordinates around X where functions ξ^μ satisfy equation (1) if and only if functions ξ^μ satisfy the usual Jacobi equation. When this is the case, it implies that by a suitable choice of coordinates, the nonlinear generalized Jacobi equation can be replaced by the linear Jacobi equation. Since the Jacobi equation transforms tensorially, this motivates a further understanding of the transformation properties of the generalized Jacobi equation. Namely:

> Suppose $X \colon I \to M$ is a geodesic in overlapping coordinates x^μ and $\widetilde{x}^\mu$. If functions $\xi^\mu \colon I \to \mathbb{R}$ solve equation (1) in coordinates x^μ, does there exist a transformation rule $\xi^\mu \mapsto \widetilde{\xi}^\mu$ such that functions $\widetilde{\xi}^\mu$ solve equation (1) in coordinates $\widetilde{x}^\mu$?

In this note we will study this question for the tensorial transformation rule $\widetilde{\xi}^\mu = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu}\xi^\nu$. We will prove two results. First, in *proposition* 4.1 we show that the tensorial transformation rule for ξ^i preserves solutions to equation (1) for affine coordinate changes. As a consequence, equation (1) is a tensorial equation when restricted to suitable Fermi coordinate systems along a geodesic (see Proposition 2.2). This motivates the use of Fermi coordinates for the study of the generalized Jacobi equation as in [CM02, Per08]. Second, in *proposition* 4.2 we explicitly show that in dimensions $n \geq 3$, the tensorial transformation rule for ξ^μ does not in general preserve solutions to equation (1). Thus, if there exists a transformation rule $\xi^\mu \to \widetilde{\xi}^\mu$ for solutions to equation (1), it is not the tensorial transformation rule.

The organization of this note is as follows. In Section 2 we review the necessary theory for affine connections and Fermi coordinates. In Section 3 we summarize the derivation of the Jacobi equation and the generalized Jacobi equation [LC27]. Lastly, in Section 4 we prove *proposition* 4.1 and *proposition* 4.2 described above.

2. Preliminaries

Let M be a smooth manifold of dimension $n \geq 2$. By TM we denote the tangent bundle with projection $\pi \colon TM \to M$. The tangent space at a point

$p \in M$ is defined as $T_pM = \pi^{-1}(p)$. Throughout the paper we denote by I an interval in $\mathbb{R}$. We will also use the Einstein summing convention.

We assume that M is endowed with an *affine connection* ∇. Thus, in each coordinate chart (U, x^μ), ∇ is represented by connection coefficients $\Gamma^\mu_{\nu\sigma}$ and if $\Gamma^\mu_{\nu\sigma}$ and $\widetilde{\Gamma}^\mu_{\nu\sigma}$ represent ∇ on overlapping coordinates x^μ and $\widetilde{x}^\mu$, we have transformation rules

$$\frac{\partial \widetilde{x}^\lambda}{\partial x^\alpha}\Gamma^\alpha_{\nu\sigma} = \widetilde{\Gamma}^\lambda_{\alpha\beta}\frac{\partial \widetilde{x}^\alpha}{\partial x^\nu}\frac{\partial \widetilde{x}^\beta}{\partial x^\sigma} + \frac{\partial^2 \widetilde{x}^\lambda}{\partial x^\nu \partial x^\sigma}. \tag{2}$$

A connection is *torsion-free* if $\Gamma^\mu_{\nu\sigma} = \Gamma^\mu_{\sigma\nu}$. If X is a curve $X\colon I \to M$, then X is an *(affinely parameterized) geodesic* if *(i)* X is a *regular curve*, that is, the tangent $\dot{X}$ is never zero, and *(ii)* in each coordinate chart (U, x^μ) that overlaps X we have

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\sigma}(X)\dot{X}^\nu\dot{X}^\sigma = 0, \tag{3}$$

where $(X^\mu(s))$ are components for X in coordinates x^μ.

2.1. **Fermi coordinates.** In this section we collect some results on Fermi coordinates. These are local coordinates for a tubular neighborhood around a geodesic [MM63]. Suppose $\mathcal{I} \subset \mathbb{R}$ is an open interval and $X\colon \mathcal{I} \to M$ is a geodesic of an affine connection on M, and suppose Π is an $(n-1)$-dimensional vector space in $T_{X(s_0)}M$ for some $s_0 \in \mathcal{I}$ such that Π is complementary to $\dot{X}(s_0)$, that is,

$$T_{X(s_0)}M = \operatorname{span}\{\dot{X}(s_0)\} \oplus \Pi. \tag{4}$$

Let $\{e_1, \ldots, e_{n-1}\}$ be a basis for Π. By parallel transport we can extend each vector $e_i \in T_{X(s_0)}M$ into a vector field $e_i(s)$ along X. Since parallel transport is a linear isomorphism, it follows that

$$T_{X(s)}M = \operatorname{span}\{\dot{X}(s), e_1(s), \ldots, e_{n-1}(s)\} \tag{5}$$

for all $s \in \mathcal{I}$. For $s \in \mathcal{I}$, let f be the map

$$f(s, z^1, \ldots, z^{n-1}) = \exp\left\{\sum_{i=1}^{n-1} z^i e_i(s)\right\} \tag{6}$$

defined for $z^1, \ldots, z^{n-1} \in \mathbb{R}$ for which the right hand side is defined.

Proposition 2.1. *Suppose $\mathcal{I}$ is an open interval, $X\colon \mathcal{I} \to M$ is a geodesic for an affine connection ∇, and $\{e_i\}_{i=1}^{n-1}$ and f are as in equation* (6)*. Moreover, suppose X has no self-intersections, and I is a proper open subset $I \subset \mathcal{I}$ such that $\overline{I}$ is compact. Then there exists an open neighborhood of the origin $B \subset \mathbb{R}^{n-1}$ such that f restricts to a diffeomorphism $f\colon I \times B \to M$ onto its range.*

Proof. Since $\overline{I}$ is compact, we can find an open ball $B \subset \mathbb{R}^{n-1}$ containing 0 such that $f\colon I \times B \to M$ is smooth. For $s \in I$ we have

$$\begin{aligned}
\frac{\partial f}{\partial s}(s, 0, \ldots, 0) &= \dot{X}(s), \\
\frac{\partial f}{\partial z^i}(s, 0, \ldots, 0) &= e_i(s), \quad i \in \{1, \ldots, n-1\}.
\end{aligned}$$

Since $\{\dot{X}(s), e_1(s), \ldots, e_{n-1}(s)\}$ are linearly independent for all $s \in I$, the inverse function theorem implies (after possibly shrinking B) that $f\colon I \times B \to M$ is a local diffeomorphism onto its range. The result follows by [Spi79, p. 345, Lemma 19]. □

When $(s, z^1, \ldots, z^{n-1}) \in I \times B$ are as in Proposition 2.1 we say that $(s, z^1, \ldots, z^{n-1})$ are *Fermi coordinates* along $X\colon I \to M$. These coordinates are determined by the geodesic $X\colon I \to M$, the initial point $X(s_0)$ and the set of vectors $\{e_i(s_0)\}_{i=1}^{n-1}$ in equation (5). One can prove that when the connection is torsion-free, then all Christoffel symbols $\Gamma^\mu_{\nu\rho}$ vanish on the central geodesic in Fermi coordinates. The next proposition shows that if two Fermi coordinates systems are determined by the same initial complementary hyperplane Π, then the Fermi coordinates differ by an affine coordinate transformation. Here, two overlapping coordinates (U, x^μ) and $(\widetilde{U}, \widetilde{x}^\mu)$ are related by an *affine coordinate transformation* if

$$\widetilde{x}^\mu(x^0, \ldots, x^{n-1}) \quad = \quad \Lambda^\mu_\nu x^\nu + C^\mu \tag{7}$$

for some constants $(\Lambda^\mu_\nu)_{\mu,\nu=0}^{n-1}$ and $(C^\mu)_{\mu=0}^{n-1}$.

Proposition 2.2. *Suppose ∇ is an affine connection on M and $X\colon I \to M$ and $\widetilde{X}\colon \widetilde{I} \to M$ are two geodesics that differ by a reparameterization. Moreover, suppose $(s, z^1, \ldots, z^{n-1})$ and $(\widetilde{s}, \widetilde{z}^1, \ldots, \widetilde{z}^{n-1})$ are Fermi coordinates along X and $\widetilde{X}$ that correspond to the same initial complementary hyperplane Π. Then Fermi coordinates $(s, z^1, \ldots, z^{n-1})$ and $(\widetilde{s}, \widetilde{z}^1, \ldots, \widetilde{z}^{n-1})$ are related by an affine coordinate transformation on their common domain. More precisely, there are constants A, B and an invertible matrix $(T^\nu_\nu)_{\mu,\nu=1}^{n-1}$ such that*

$$\widetilde{s} = As + B, \quad \widetilde{z}^\mu = \sum_{\nu=1}^{n-1} T^\mu_\nu z^\nu \text{ for } \mu \in \{1, \ldots, n-1\}. \tag{8}$$

Proof. Let f and $\widetilde{f}$ be maps $f\colon I \times B \to M$ and $\widetilde{f}\colon \widetilde{I} \times \widetilde{B} \to M$ as in equation (6) that define the two Fermi coordinates. Let $\psi\colon I \to \widetilde{I}$ be the reparameterization, so that $\widetilde{X} \circ \psi = X$. Writing out the geodesic equation for $\widetilde{X}$ and $\widetilde{X} \circ \psi$ shows that $\psi(s) = As + B$ for some $A \in \mathbb{R} \setminus \{0\}$ and $B \in \mathbb{R}$. By assumption, there are $s_0 \in I$ and $\widetilde{s}_0 \in \widetilde{I}$ such that vectors $\{\frac{\partial}{\partial z^\mu}|_{X(s_0)}\}_{\mu=1}^{n-1}$ and $\{\frac{\partial}{\partial \widetilde{z}^\mu}|_{\widetilde{X}(\widetilde{s}_0)}\}_{\mu=1}^{n-1}$ span the same hyperplane. Thus $X(s_0) = \widetilde{X}(\widetilde{s}_0)$ and there exist an invertible matrix $T = (T^\nu_\mu)_{\mu,\nu=1}^{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$\left.\frac{\partial}{\partial z^\mu}\right|_{X(s)} \quad = \quad \sum_{\nu=1}^{n-1} T^\nu_\mu \left.\frac{\partial}{\partial \widetilde{z}^\nu}\right|_{X(s)}, \quad \mu \in \{1, \ldots, n-1\} \tag{9}$$

at $s = s_0$. Since both sides in this equation are parallel vectors along X, and since equality holds for $s = s_0$, it follows that equation (9) holds for all $s \in I$.

Suppose $(s, z^1, \ldots, z^{n-1})$ and $(\tilde{s}, \widetilde{z}^1, \ldots, \widetilde{z}^{n-1})$ are Fermi coordinates for the same point, so that

$$f(s, z^1, \ldots, z^{n-1}) \quad = \quad \widetilde{f}(\widetilde{s}, \widetilde{z}^1, \ldots, \widetilde{z}^{n-1}). \tag{10}$$

Then equations (6) and (9) imply that

$$f(s, z^1, \dots, z^{n-1}) = \widetilde{f}\left(As + B, \sum_{\mu=1}^{n-1} T^1_\mu z^\mu, \dots, \sum_{\mu=1}^{n-1} T^{n-1}_\mu z^\mu\right). \tag{11}$$

Since f is a bijection onto its range, equations (10)–(11) imply that equation (8) holds. □

Let us consider the case when (M, g) is a pseudo-Riemann manifold of index 1 and ∇ is the Levi-Civita connection of g. If $\dot{X}(s_0)$ is timelike or spacelike, then a suitable hyperplane Π is given by $\Pi = (\dot{X}(s_0))^\perp$ [O'N83, p. 49]. In this case, the metric can further be made diagonal along the geodesic in Fermi coordinates [LC27]. On the other hand, if $\dot{X}(s_0)$ is lightlike, we have $\dot{X}(s_0) \in (\dot{X}(s_0))^\perp$ and the choice $\Pi = (\dot{X}(s_0))^\perp$ is not possible.

3. Equations of geodesic deviation

In this section we describe three equations for the behavior of nearby geodesics: *the exact deviation equation*, *the Jacobi equation*, and *the generalized Jacobi equation*. Throughout this section we assume that ∇ is an affine and torsion-free connection on M.

3.1. **The exact geodesic deviation equation.** Suppose $x\colon I \to M$ and $X\colon I \to M$ are two geodesics that are contained in one coordinate chart (U, x^μ). If locally $x(s) = (x^\mu(s))$ and $X(s) = (X^\mu(s))$ for $s \in I$, let $\xi\colon I \to \mathbb{R}^n$ be the displacement between the geodesics defined as

$$\xi^\mu(s) = x^\mu(s) - X^\mu(s), \quad s \in I. \tag{12}$$

Since x and X are solutions to the geodesic equation, it follows that

$$\ddot{\xi}^\mu + \Gamma^\mu_{\nu\sigma}(X + \xi)\left(\dot{X}^\nu + \dot{\xi}^\nu\right)\left(\dot{X}^\sigma + \dot{\xi}^\sigma\right) - \Gamma^\mu_{\nu\sigma}(X)\,\dot{X}^\sigma\,\dot{X}^\nu = 0. \tag{13}$$

We will refer to (13) as the *exact geodesic deviation equation* (see [Per08]). Equation (13) is an exact equation in the sense that its derivation does not involve any approximations. However, the geometric analysis of equations (12)–(13) becomes difficult since both equations are defined in local coordinates and, moreover, both equations involve two points on the manifold.

That components ξ^μ defined by equation (12) do not define a vector field along $X\colon I \to M$ can be seen as follows. If $\mathbb{R}^2$ is equipped with the Euclidean metric and Cartesian coordinates (x^1, x^2), then curves $X(s) = (s, 0)$ and $x(s) = (s, 1)$ for $s > 0$ are geodesics and functions $\xi^\mu\colon I \to \mathbb{R}^2$ are given by $\xi(s) = (0, 1)$. However, in polar coordinates $\widetilde{x}^1 = r, \widetilde{x}^2 = \theta$, the definition of ξ yields $\widetilde{\xi}(s) = (\sqrt{1+s^2} - s, \tan^{-1}(\frac{1}{s}))$. It follows that $\xi^1(s) = \left(\frac{\partial x^1}{\partial \widetilde{x}^\mu}\right)\big|_{X(s)} \widetilde{\xi}^\mu$ is not satisfied for all $s > 0$, and functions ξ^μ in equation (12) do not in general transformation as a tensor.

Even if functions $\xi^\mu(s)$ in equation (12) do not transform as a tensor in general, it turns out that if we restrict to suitable Fermi coordinates along a geodesic, then functions $\xi^0, \dots, \xi^{n-1}$ transform as a vector. To see this,

suppose $X\colon I \to M$ is a geodesic and $(s, z^1, \dots, z^{n-1})$ are Fermi coordinates along X as in *proposition* 2.1. If $x\colon I \to M$ is another geodesic that can be written as $x(s) = (s, z^1(s), \dots, z^{n-1}(s))$ in these Fermi coordinates, then functions $\xi^0, \dots, \xi^{n-1}$ in equation (12) are given by

$$\xi^\mu(s) \quad = \quad \begin{cases} 0, & \text{for } \mu = 0, \\ z^\mu(s), & \text{for } \mu \in \{1, \dots, n-1\}. \end{cases} \tag{14}$$

If $(\widetilde{s}, \widetilde{z}^1, \dots, \widetilde{z}^{n-1})$ are other Fermi coordinates along X as in *proposition* 2.2 and $\widetilde{\xi}^\mu$ are defined by equation (12) in these coordinates, then equations (6) and (8) show that ξ^μ and $\widetilde{\xi}^\mu$ transform as a vector along X.

Since we have not made any approximations when deriving equation (13), it turns out that we can transform solutions from one coordinate system to another. However, this will lead to a highly non-standard transformation rule. To see this, suppose $X^\mu(s)$ and $x^\mu(s)$ are geodesics as above in coordinates x^μ whence $\xi^\mu(s)$ in equation (12) solve equation (13). Thus, if $\widetilde{x}^\mu$ are overlapping coordinates and $\widetilde{X}^\mu(s)$ and $\widetilde{x}^\mu(s)$ represent the same geodesics in these coordinates, then functions $\widetilde{\xi}^\mu(s) = \widetilde{x}^\mu(s) - \widetilde{X}^\mu(s)$ solve equation (13) in $\widetilde{x}^\mu$-coordinates. Then

$$\widetilde{\xi}^\mu(s) \quad = \quad (\widetilde{x} \circ x^{-1})^\mu \left(\xi^\lambda(s) + X^\lambda(s) \right) - \widetilde{X}^\mu(s), \tag{15}$$

and the above equation gives a transformation rule $\xi^\mu \mapsto \widetilde{\xi}^\mu$ for solutions to the exact deviation equation along a geodesic $X\colon I \to M$.

3.2. **The Jacobi equation.**

The usual approach to analyze the qualitative behavior of nearby geodesics is by using the *Jacobi equation*, which describes the displacement ξ^μ between two geodesics under the assumption that ξ^μ and $\dot{\xi}^\mu$ are infinitesimal [LC27].

Let us show how the Jacobi equation follows from equation (13). Using Taylor's formula, we can expand each connection coefficients as

$$\Gamma^\mu_{\nu\sigma}(X + \xi) \quad = \quad \Gamma^\mu_{\nu\sigma}(X) + \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\tau}(X) \xi^\tau + \text{higher order terms.} \tag{16}$$

Inserting equation (16) into equation (13) and assuming that ξ^μ, $\dot{\xi}^\mu$ are infinitesimal yields

$$\ddot{\xi}^\mu + \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\tau}(X)\, \xi^\tau \dot{X}^\nu \dot{X}^\sigma + 2\,\Gamma^\mu_{\nu\sigma}(X)\, \dot{X}^\sigma \dot{\xi}^\nu \quad = \quad 0. \tag{17}$$

That is, in the above we assume that functions ξ^μ are such that all higher order terms $\xi^\mu \xi^\nu$, $\dot{\xi}^\mu \xi^\nu$, $\dot{\xi}^\mu \dot{\xi}^\nu$, $\xi^\mu \xi^\nu \xi^\sigma$, $\dot{\xi}^\mu \xi^\nu \xi^\sigma$, $\dots$ can be neglected. Let us emphasize that due to this assumption on ξ^μ, we can no longer treat ξ in equation (17) as the exact displacement between two geodesics as in equation (12).

Equation (17) is known as the *Jacobi equation* for an affine and torsion-free connection ∇. Using the covariant derivative $\frac{D}{Ds}$ along X, equation (17) can equivalently be rewritten as

$$\frac{D^2}{Ds^2} \xi + R(\xi, \dot{X})(\dot{X}) \quad = \quad 0, \tag{18}$$

where $R(\xi, \dot{X})$ the curvature endomorphism of ∇ and $\xi(s) = \xi^\mu(s)\frac{\partial}{\partial x^\mu}|_{X(s)}$. See for example [Per08]. From equations (17)–(18) we see that the assumptions on ξ^μ and $\dot{\xi}^\mu$ have simplified the exact geodesic deviation equation in three significant ways: First, unlike the exact geodesic deviation equation, equations (17) and (18) are linear equations in functions ξ^μ. Second, as we will see below, equations (17) and (18) will be covariant when functions ξ^μ transform as components of a tensor. This is a simplification when compared with the nonlinear transformation rule (15) for solutions to the exact geodesic deviation equation. Lastly, equations (17) and (18) are equations along X that only involve evaluations at $X(s)$ on M.

In the next sections we will study the coordinate invariance of the generalized Jacobi equation. As a model for this analysis and to fix notation, let us consider in some detail the coordinate invariance for the Jacobi equation. Suppose (U, x^μ) are local coordinates for M and $\xi = \xi^\mu \frac{\partial}{\partial x^\mu}\big|_{X(s)}$ is a curve $\xi\colon I \to TU$ along a geodesic $X\colon I \to M$. For $\mu \in \{0, \dots, n-1\}$ we define

$$J^\mu_U[(\xi^\lambda)_{\lambda=0}^{n-1}] \quad = \quad \ddot{\xi}^\mu + \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\tau}\,\xi^\tau \dot{X}^\nu \dot{X}^\sigma + 2\,\Gamma^\mu_{\nu\sigma}\,\dot{X}^\sigma \dot{\xi}^\nu. \tag{19}$$

That is, the right hand side is the differential operators that appears in the Jacobi equation. To simplify the notation we will also write $J^\mu_U[(\xi^\lambda)_{\lambda=0}^{n-1}] = J^\mu_U[\xi^\lambda]$. We say that a curve $\xi\colon I \to TU$ is a *Jacobi field* in U if $\pi \circ \xi\colon I \to U$ is a geodesic and $J^\mu_U[\xi^\lambda] = 0$ for all $\mu \in \{0, \dots, n-1\}$.

If $(\widetilde{U}, \widetilde{x}^\mu)$ are overlapping coordinates then

$$J^\mu_{\widetilde{U}}\left[\frac{\partial \widetilde{x}^\lambda}{\partial x^\rho}\xi^\rho\right] \quad = \quad \frac{\partial \widetilde{x}^\mu}{\partial x^\sigma} J^\sigma_U\left[\xi^\lambda\right]. \tag{20}$$

Thus, if we assume that ξ^μ are components for a vector field along X, then the definition of a Jacobi field does not depend on the choice of coordinates.

There are also other ways to derive the Jacobi equation. One approach is to start with a *geodesic variation* around a geodesic $X\colon I \to M$. That is, a map $\Lambda\colon (-\varepsilon_0, \varepsilon_0) \times I \to M$ such that

(i) $s \mapsto \Lambda(\varepsilon, s)$ is a geodesic for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$,
(ii) $\Lambda(0, s) = X(s)$.

Then one can show that the tangent of Λ in the ε-direction defines a vector field $\xi\colon I \to TM$, $\xi(s) = \partial_\varepsilon \Lambda(\varepsilon, s)|_{\varepsilon=0}$ along X, and moreover, the components of ξ solves the Jacobi equation (17). Conversely, any solutions to the Jacobi equation on a compact interval, can be written as $\xi(s) = \partial_\varepsilon \Lambda(\varepsilon, s)|_{\varepsilon=0}$ for a geodesic variation Λ. See for example [BD10]. An advantage of this derivation is that ξ a tangent vector by definition, and the derivation does not involve any assumptions like ξ^μ and $\dot{\xi}^\mu$ beeing small or infinitesimal.

3.3. **The generalized Jacobi equation.**

In the previous section we started with the exact geodesic deviation equation (13), did a Taylor expansion of $\Gamma^\mu_{\nu\sigma}(X + \xi)$ (equation (16)) and assumed that ξ^μ and $\dot{\xi}^\mu$ are infinitesimal, so that higher order terms $\xi^\mu\xi^\nu$, $\dot{\xi}^\mu\xi^\nu$, $\dot{\xi}^\mu\dot{\xi}^\nu$, $\xi^\mu\xi^\nu\xi^\sigma$, $\dot{\xi}^\mu\xi^\nu\xi^\sigma$, $\dot{\xi}^\mu\dot{\xi}^\nu\xi^\sigma$, $\dots$ can be neglected. This gives rise to the Jacobi equation (17). The generalized

Jacobi equation is derived in the same way, but assuming that only ξ^μ is infinitesimal. Under this approximation, equation (13) simplifies into the generalized Jacobi equation in equation (1).

Suppose (U, x^μ) are local coordinates for M, and $\xi^0, \dots, \xi^{n-1} \colon I \to \mathbb{R}$ are functions along a geodesic $X \colon I \to U$. As for the Jacobi equation we define differential operators

$$G^\mu_U[\xi^\lambda] \;=\; \ddot{\xi}^\mu + \Gamma^\mu_{\nu\sigma}\left(2\dot{\xi}^\nu \dot{X}^\sigma + \dot{\xi}^\nu\dot{\xi}^\sigma\right) + \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\alpha}\xi^\alpha \left(\dot{X}^\nu + \dot{\xi}^\nu\right)\left(\dot{X}^\sigma + \dot{\xi}^\sigma\right),$$

for $\mu \in \{0, \dots, n-1\}$. We will say that functions $\xi^0 \dots, \xi^{n-1} \colon I \to \mathbb{R}$ *define a generalized Jacobi field in chart U* if $G^\mu_U[\xi^\lambda] = 0$ for $\mu \in \{0, \dots, n-1\}$.

Suppose X is a geodesic $X \colon I \to M$. Then the Jacobi equation along X is locally a linear second order differential equation for components ξ^μ. Thus, for initial values $\xi^\mu(s_0)$ and $\dot{\xi}^\mu(s_0)$ there exists a unique Jacobi field $\xi \colon I \to TM$ along X with these initial values. See for example [Lee97]. In contrast, the generalized Jacobi equation $G^\mu_U[\xi^\lambda] = 0$ is locally a *non-linear* second order differential equation for functions ξ^μ. The Picard-Lindelöf theorem implies that for any $s_0 \in I$, there exists an neighborhood around s_0 such that the generalized Jacobi is uniquely solvable on this neighborhood from initial values $\xi^\mu(s_0)$ and $\dot{\xi}^\mu(s_0)$. One can bound the size of this neighborhood [Har64, Theorem 1.1, Chapter 2]. However, in general there is no guarantee that a unique solution exists on the entire interval I as for Jacobi fields.

For future reference, let us note that differential operators $G^\mu_U[\xi^\lambda]$ and $J^\mu_U[\xi^\lambda]$ are related by

$$G^\mu_U[\xi^\lambda] \;=\; J^\mu_U[\xi^\lambda] + \Delta^\mu_U[\xi^\lambda], \tag{21}$$

where

$$\Delta^\mu_U[\xi^\lambda] \;=\; \Gamma^\mu_{\nu\sigma}\,\dot{\xi}^\nu\dot{\xi}^\sigma + 2\frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\alpha}\,\xi^\alpha \dot{X}^\nu \dot{\xi}^\sigma + \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\alpha}\,\xi^\alpha \dot{\xi}^\nu\dot{\xi}^\sigma. \tag{22}$$

4. Coordinate invariance of the generalized Jacobi equation

As described in the introduction, this section contains the main results of this note. As in Section 3 we assume that M is a manifold with an affine and torsion-free connection ∇.

Proposition 4.1. *Suppose $X \colon I \to M$ is a geodesic contained in a chart (U, x^μ) and functions $\xi^0, \dots, \xi^{n-1} \colon I \to \mathbb{R}$ define a generalized Jacobi field in coordinates x^μ. If $\widetilde{x}^\mu$ are coordinates defined by the affine coordinate transformation* (7)*, then functions $\frac{\partial \widetilde{x}^\mu}{\partial x^\lambda}\xi^\lambda \colon I \to \mathbb{R}$ define a generalized Jacobi field in coordinates $\widetilde{x}^\mu$.*

Proof. Let $\widetilde{\xi}^\mu = \frac{\partial \widetilde{x}^\mu}{\partial x^\sigma}\xi^\sigma$. Equations (23)–(25) in Appendix A show that $\frac{\partial \widetilde{x}^\mu}{\partial x^\nu}\Delta^\nu_U[\xi^\lambda] = \Delta^\mu_{\widetilde{U}}[\widetilde{\xi}^\lambda]$. By equations (20) and (21) we then have $\frac{\partial \widetilde{x}^\mu}{\partial x^\nu} G^\nu_U[\xi^\lambda] = G^\mu_{\widetilde{U}}[\widetilde{\xi}^\lambda]$ and the claim follows. □

Proposition 4.2. *Suppose M is a manifold of dimension ≥ 3, $X\colon I \to M$ is a geodesic and (U, x^μ) are local coordinates around $X(s_0)$ for some $s_0 \in I$. By shrinking I to a neighborhood of s_0 we can find functions $\xi^\mu\colon I \to \mathbb{R}$ and overlapping coordinates $(\widetilde{U}, \widetilde{x}^\mu)$ around $X(s_0)$ such that ξ^μ define a generalized Jacobi field in chart U, but $\frac{\partial \widetilde{x}^\mu}{\partial x^\nu}\xi^\nu$ does not define a generalized Jacobi field in chart $\widetilde{U}$.*

Proof. Let $p = X(s_0)$. By *proposition* 4.1 we may assume that $0 \in \mathbb{R}^n$ corresponds to p in coordinates x^μ. Let $(\widetilde{U}, \widetilde{x}^\mu)$ be the coordinates determined by

$$\widetilde{x}^\mu(x^0, \ldots, x^{n-1}) = x^\mu + \frac{1}{6} T^\mu_{\tau\rho\sigma} x^\tau x^\rho x^\sigma,$$

where

$$T^\mu_{\tau\rho\sigma} = \delta^\mu_\tau \delta_{\rho\sigma} + \delta^\mu_\rho \delta_{\tau\sigma} + \delta^\mu_\sigma \delta_{\tau\rho},$$

and δ^μ_ν and $\delta_{\mu\nu}$ are the Kronecker delta symbols.

Let $u = \dot{X}(s_0)$. Since $\dim M \geq 3$ we can find vectors $v, w \in T_pM$ such that $\{u, v, w\}$ is an orthonormal basis with respect to the Euclidean metric $g = \delta_{\mu\nu} dx^\mu \otimes dx^\nu$ on U. By the Picard-Lindelöf theorem we can shrink I to a neighborhood of s_0 and find functions $\xi^\mu\colon I \to \mathbb{R}$ such that ξ^μ define a generalized Jacobi field in chart U and

$$\dot{X}^\mu(s_0) = u^\mu, \quad \xi^{\,\mu}(s_0) = v^\mu, \quad \dot{\xi}^{\,\mu}(s_0) = w^\mu,$$

when $u = u^\mu \frac{\partial}{\partial x^\mu}|_p$, $v = v^\mu \frac{\partial}{\partial x^\mu}|_p$ and $w = w^\mu \frac{\partial}{\partial x^\mu}|_p$. Let $\widetilde{\xi}^\mu = \frac{\partial \widetilde{x}^\mu}{\partial x^\sigma}\xi^\sigma$.

Contracting $G^\nu_{\widetilde{U}}[\widetilde{\xi}^\lambda] = J^\nu_{\widetilde{U}}[\widetilde{\xi}^\lambda] + \Delta^\nu_{\widetilde{U}}[\widetilde{\xi}^\lambda]$ by $\frac{\partial x^\mu}{\partial \widetilde{x}^\nu}$, applying equations (20)–(21) and equations (23)–(25) in Appendix A yields

$$\begin{aligned}
\frac{\partial x^\mu}{\partial \widetilde{x}^\nu} G^\nu_{\widetilde{U}}[\widetilde{\xi}^\lambda] &= J^\mu_U[\xi^\lambda] + \frac{\partial x^\mu}{\partial \widetilde{x}^\nu} \Delta^\nu_{\widetilde{U}}[\widetilde{\xi}^\lambda] \\
&= \frac{\partial x^\mu}{\partial \widetilde{x}^\nu} \Delta^\nu_{\widetilde{U}}[\widetilde{\xi}^\lambda] - \Delta^\mu_U[\xi^\lambda] \\
&= -\frac{\partial x^\mu}{\partial \widetilde{x}^\nu} \frac{\partial^3 \widetilde{x}^\nu}{\partial x^\tau \partial x^\rho \partial x^\sigma} \xi^\tau \left(2\dot{X}^{\,\rho} + \dot{\xi}^{\,\rho}\right) \dot{\xi}^{\,\sigma},
\end{aligned}$$

where all expressions are evaluated at s_0. Since $T^\mu_{\tau\rho\sigma}$ is symmetric in $\tau\rho\sigma$, we have $\frac{\partial^3 \widetilde{x}^\mu}{\partial x^\tau \partial x^\rho \partial x^\sigma} = T^\mu_{\tau\rho\sigma}$. Moreover, since $\frac{\partial x^\mu}{\partial \widetilde{x}^\nu}(p) = \delta^\mu_\nu$ and since u, v, w are orthonormal it follows that

$$\begin{aligned}
G^\mu_{\widetilde{U}}[\widetilde{\xi}^\lambda] &= -g(2u + w, w)\, v^\mu - g(v, w)\,(2u^\mu + w^\mu) - g(v, 2u + w)\, w^\mu \\
&= -v^\mu
\end{aligned}$$

at s_0. We have shown that $G^\mu_{\widetilde{U}}[\widetilde{\xi}^\lambda] \neq 0$ at s_0. Thus functions $\widetilde{\xi}^\mu$ do not define a generalized Jacobi field in chart $\widetilde{U}$, and the claim follows. □

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Appendix A. Coordinate transformations

Suppose X is a curve $X\colon I \to M$, and x^μ and $\widetilde{x}^\mu$ are coordinates around $X(s_0)$ with $\frac{\partial^2 \widetilde{x}^\mu}{\partial x^\nu \partial x^\rho}(X(s_0)) = 0$ for some $s_0 \in I$. If ξ^μ are functions $\xi^\mu\colon I \to \mathbb{R}$ and functions $\widetilde{\xi}^\mu$ are defined by $\widetilde{\xi}^\mu = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu}\xi^\nu$, then at $X(s_0)$ we have transformation rules

$$\dot{\xi}^\mu = \frac{\partial x^\mu}{\partial \widetilde{x}^\nu}\dot{\widetilde{\xi}}^\nu, \tag{23}$$

$$\frac{\partial \widetilde{x}^\mu}{\partial x^\sigma}\Gamma^\sigma_{\rho\nu} = \widetilde{\Gamma}^\mu_{\sigma\lambda}\frac{\partial \widetilde{x}^\sigma}{\partial x^\rho}\frac{\partial \widetilde{x}^\lambda}{\partial x^\nu}, \tag{24}$$

$$\frac{\partial \widetilde{x}^\mu}{\partial x^\sigma}\frac{\partial \Gamma^\sigma_{\nu\rho}}{\partial x^\lambda} = \frac{\partial \widetilde{\Gamma}^\mu_{\epsilon\sigma}}{\partial \widetilde{x}^\delta}\frac{\partial \widetilde{x}^\delta}{\partial x^\lambda}\frac{\partial \widetilde{x}^\sigma}{\partial x^\nu}\frac{\partial \widetilde{x}^\epsilon}{\partial x^\rho} + \frac{\partial^3 \widetilde{x}^\mu}{\partial x^\nu \partial x^\rho \partial x^\lambda}. \tag{25}$$

These equations show that between coordinates x^μ and $\widetilde{x}^\mu$, objects $\xi^\mu, \dot{\xi}^\mu$, $\Gamma^\mu_{\rho\nu}$ and $\frac{\partial \Gamma^\mu_{\rho\nu}}{\partial x^\sigma}$ transform as tensors at $X(s_0)$ (up to an extra term in the last equation). For a general coordinate transformation, these transformation rules are considerably more involved. See equation (2).

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